Private Provision of a Complementary Public Good*

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Preliminary Version

Abstract

For several years, an increasing number of firms are investing in Open Source Software (OSS). While improvements in such a non-excludable public good are not appropriable, companies can benefit indirectly in a complementary proprietary segment. We study this incentive for investment in OSS. In particular we ask how (1) market entry and (2) public investments in the public good affects the firms’ behaviour and profits. Surprisingly, we find that there exist cases where incumbents benefit from market entry. Moreover, we show the counter-intuitive result that public spending does not necessarily lead to a decreasing voluntary private contribution.

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1 Introduction

For several years, an increasing number of firms like IBM and Hewlett-Packard or Suse and Red Hat have begun to invest in Open Source Software. Open Source Software, such as Linux, is typically under the General Public License (GPL) and therefore any improvement must be provided for free. Hence, an Open Source Software can be seen as a non-excludable public good meaning that firms are not able to sell the Open Source Software or their improvements. This issue raises the question why companies do contribute to such a public good.

Lerner and Tirole (2000) argue that firms expect to benefit from their expertise in some market segment of which the demand is boosted by the introduction of a complementary Open Source Software. Although the companies cannot capture directly the value of an open source program’s improvement, they can profit indirectly through selling more complementary proprietary goods at a potentially higher price.

Notice, that this incentive to contribute to a non-excludable public good arises not only in the case of Open Source Software. In fact, if e.g. a firm’s advertising increases not only the demand for the firm’s own good but also the demand for its competitors’ products (Friedman (1983) calls it cooperative advertising) or if a firm’s lobby-activity has a positive effect on the whole industry, the analysis remains the same.

In this paper we study the incentive of investment in such a non-excludable public good. In particular we address the following two questions:

(1) What is the effect of a higher public good investment by the government on the firms’ production decisions and their profits?

(2) How does market entry and therefore fiercer competition affect the incentive to contribute to the public good and how does it influence the incumbents’ profits?

We contribute to answering the questions (1) and (2) by analyzing a model with Cournot-Competition where the firms can produce a private and a public good, but can sell only the private good. For the consumers, the private and public goods are complements. An increase in the available quantity of the public good increases their willingness to pay for the private good.
The first question is especially interesting because of the ongoing discussion if and how the government should support Open Source Software. One might fear that an increase in the government’s contribution to the public good decreases firms’ voluntary spending like it is known from the public good literature (e.g. Bergstrom et al. (1985)). Interestingly, we will show that this does not have to be the case and that it is even possible that the firms’ investments increase if the government increases its contribution to the public good. Thus it is not obvious whether the government’s and the firms’ public good investments are strategic complements or substitutes to each other.

The second question is shortly addressed by Lerner and Tirole (2000). They argue that the usual free-rider problem should appear because the firms are not able to capture all the benefits of their investments. Therefore, one might think that with an increasing number of firms the free-rider problem gets worse and so the firms’ contribution to the public good decrease. As a consequence the firms’ profits and the social surplus might decrease.

We show in this paper that the market entry of an additional firm has a positive externality (through the entrant’s contribution to the public good) and a negative externality (through the entrant’s production of the private good) on the incumbents. We find that for certain cost and demand functions each firm reduces its output as a consequence of market entry and suffers a decrease in profits. In this case incumbents dislike market entry. Surprisingly, for certain cost and demand constellations, it is also possible that with market entry every firm expands its output and is able to increase its profit. However, a social planner unambiguously prefers market entry.

This article is related to the public good literature that is concerned with the private provision of a non-excludable public good. In standard models of public good provision, households can buy or produce the private and the public good and they receive a certain utility directly from the existence of these goods.

In our model, firms do not benefit directly form the production of a public good.

2See e.g. Bergstrom et al. (1986) for a general approach and e.g Bitzer and Schröder (2002) or Johnson (2002) for an application to Open Source Software.
They produce the public good because of the complementarity to the private good. This leads to a new effect: The incentive of the firms to contribute to the public good depends on the market environment and therefore on the number of competitors. In the normal setup the marginal utility of the public good is determined through the utility function and is exogenously given. In our setup it is endogenous and depends on the ability to use the additional unit of the public good to earn money in the proprietary sector. This, however, depends on the degree of competition in this sector.

A second strand of literature our paper is related to is the literature of Multi-market Oligopoly. Bulow, Geanakoplos and Klemperer (1985) analyze the effects of a change in one market environment. In their model different markets are related through the cost function. In our model the markets are not related through the production technology but through the demand function. Bulow, Geanakoplos and Klemperer (1985) address this issue but do not formalize it. They mention that firms must take care off cross-effects in making marginal revenue calculations and consider the strategic effects of their actions in one market on competitors’ actions in a second. In our model we formalize this issue and extend it to the interesting case of a non-excludable public good which is in contrast to the model of Bulow et al. (1985) where only private goods are considered.

Becker and Murphy (1993) analyze a model in which advertisement and a advertised good enters the utility function of the households. Advertisement has the property that it raises the willingness to pay for the advertised good and is from the economic viewpoint complementary to the advertised good. Nevertheless, in their setup a firm’s advertisement is only complementary to its own private good. In our model it is also complementary to the competitors’ private goods.

In some sense, the paper most closely related to ours is Friedman (1983). He uses a dynamic setup and treats advertising like capital in the sense that a firm can build up a ”goodwill stock” through advertising comparable to a capital stock. Due to his modelling an increasing number of active firms leads to a steady-state in which conventional competitive effects dominate and prices converge to marginal costs. However, we look at a static game to concentrate explicitly on the externalities between the firms and the goods and come up with different results.

We will proceed as follows. The next section sets up the general model. In
Section 3 we look at the properties of the market equilibrium. After this we analyze in Section 4 the general production adjustment process and the reaction of the profits due to an exogenous change in the production of the private and the public good. In Section 5 we apply this analysis to determine the effects of a government intervention and the effects of market entry. The final section concludes.

2 The Model

Consumers

Consider individuals who consume two goods: A private good and a non-excludable public good. Let $X$ describe the available quantity of the private good, e.g. computer hardware, and $Y$ the available quantity of the non-excludable public good, e.g. Open Source Software. These two types of goods are complements for the consumers. The consumers have to pay a price $p$ for every unit of the private good they want to consume. For the non-excludable public good there is by definition no price to pay. Therefore, every individual consumes the whole available quantity of the public good.

Firms

Firm $i$ ($i \in \{1, 2, ..., N\}$) can produce the private good and the public good at costs of $K^x_i(x)$ and $K^y_i(y)$ with $x = (x_1, ..., x_N)$ and $y = (y_1, ..., y_N)$. Firm $i$’s private good production is denoted by $x_i$ and its production of the public good by $y_i$. Firm $i$’s revenue is $R_i = x_i \cdot p(X, Y)$ and its profit function is $\pi_i = R_i(X, Y) - K^x_i(x) - K^y_i(y)$.

Assumptions

\[^3\text{We are always going to speak about quantity of the public good. In some cases one can interpret this quantity as a measure of quality.}\]
In order to analyze the described questions we introduce the following assumptions:

(A1) \( x_i \in X_i = [0; \bar{X}_i], \ y_i \in Y_i = [0; \bar{Y}_i] \) and \( s_i \in S_i = X_i \times Y_i \)

(A2) \( X = \sum_{i=1}^{N_i} x_i \) and \( Y = \sum_{i=1}^{N_i} y_i \)

(A3) \( K^x_i(x) \) is convex in \( x_i \) and twice continuously differentiable with \( K^x_i(X = 0) = 0 \)

(A4) \( K^y_i(y) \) is convex in \( y_i \) and twice continuously differentiable with \( K^y_i(Y = 0) = 0 \)

(A5) The inverse demand function \( p \) is twice continuously differentiable and has the properties: \( p = p(X; Y), \ \frac{\partial p}{\partial X} \leq 0, \ \frac{\partial p}{\partial Y} \geq 0 \) and \( p = p(X = 0; Y = 0) > 0 \)

(A6) \( R_i \) is twice continuously differentiable with \( \frac{\partial^2 R_i}{\partial x_i \partial x_i} < 0, \ \frac{\partial^2 R_i}{\partial x_i \partial X} \leq 0, \ \frac{\partial^2 R_i}{\partial y_i \partial Y} \leq 0 \)

(A7) \( \left[ \frac{\partial^2 R_i}{\partial x_i \partial y_j} \right]^2 < \left[ \frac{\partial^2 K^x_i}{\partial x_i^2} - \frac{\partial^2 K^y_i}{\partial y_i^2} \right] * \left[ \frac{\partial^2 R_i}{\partial y_i^2} - \frac{\partial^2 K^y_i}{\partial y_i^2} \right] \)

Assumption (A1) restricts the strategy space \( S_i \) of a firm \( i \) to a nonempty convex subset of \( R^2 \). Therefore, the firms face a capacity constraint through e.g. implicitly assuming infinite production costs for \( s_i \notin S_i \). Nevertheless, this assumption is without loss of generality because we always let \( X_i \) and \( Y_i \) to be big enough ensuring that a possible interior solution is achievable.

Assumption (A2) states the fact that only the firms’ current production can be consumed.

Assumptions (A3) and (A4) state that the firms’ cost functions are convex in their own production decision. Furthermore, we assume that the production costs for a small quantity of \( X \) and \( Y \) are very low. In the next section we will describe the properties of the cost functions in detail.

Assumption (A5) determines the properties of the inverse demand function. It is as usual non-increasing in \( X \) because the marginal utility of the private good is declining or constant. Since the public and private goods are complements, the consumers’ willingness to pay for the private good is non-decreasing in \( Y \). For an illustration think about a computer server (= the private good) and a open source operating system (= the public good). The performance of the server depends crucially on the ability of the server operating system to use the power of the hardware. If the quality of the operating system is increasing without generating
any costs for the consumers, their willingness to pay for the server increases due to the better performance or remains constant but never falls. Furthermore, we assume in Assumption (A5) that for a certain range of $X$ and $Y$ there is a positive willingness to pay for the private good. Together with the Assumptions (A3) and (A4) this ensures an incentive to produce. Assumption (A6) states that a firm’s marginal revenue of the private good is non-increasing in the others private good production and decreasing in its own private good production. The firm’s marginal revenue of the public good is non-increasing in its own and the others public good production. Furthermore, we assume that the marginal revenue of a firm’s good increases with the firm’s production level of the complementary good. One needs this assumption for the second order conditions. Furthermore we avoid having the possibility of a strategic complementarity in respect of the firms’ private good production or public good production which is a quite natural assumption for a Cournot game. In addition we assume that the cross derivative of the price is small enough not to dominate the first derivative of the price for the marginal revenue. This ensures that the firms marginal revenue of the private good is non-decreasing with an increase in the public good production. Assumption (A7) ensures that the firm’s profit function is always concave and therefore has a unique maximum.

**Time Structure**

We assume that the firms are engaged in a one-period Cournot-Competition. Therefore, all firms decide simultaneously how much they produce of the private and the public good given the production decision of the others.

### 3 Properties of the Market Equilibrium

To determine the market equilibrium one has to think more precisely about the cost functions. For the private good like e.g computer hardware, one can easily make standard assumptions and use a convex cost function which is independent of the production decision of the others.
\[
\frac{\partial K^y_i(x_i)}{\partial x_i} \geq 0, \quad \frac{\partial^2 K^y_i(x_i)}{\partial x_i^2} \geq 0
\]

The determination of the public good cost function properties is not so straightforward. On the one hand, one can argue that every firm has the same abilities and knowledge to improve an Open Source Software. This would imply that the costs for an additional improvement of the software are equal between the firms. For example, imagine that one firm already eliminated the easiest failures in a program. If all firms have the same abilities, this leads to the fact that the elimination of an additional failure would cause the same costs for every type of proceeding firm.

Translated into the cost function, this implies that the overall costs for the public good \( K^y(Y) \) depend only on the total production \( Y \) and is independent of the production distribution between these firms.\(^4\) Nevertheless, one has to decide how to split up these costs.

In the case of \( N \) firms one possible assumption is that every firm has to bear the fraction \( \frac{1}{N} \) of the total costs.\(^5\) This would lead to a firm’s cost function where its costs depend on the production of the other firms

\[
K^y_i(Y) = \frac{K^y_i(y_i + Y_{-i})}{N},
\]

whereby \( Y_{-i} = \sum_{j=1, i \neq j}^N y_j \).

On the other hand one can bring forward the argument that the abilities and the knowledge of the firms are different. For example, for Hewlett-Packard it may be very easy to improve a certain function of Linux dealing with printing which would in contrast be very demanding for IBM and the other way round. This implies that a firm’s cost function is independent of the other firms’ production

\[
K^y_i(y) = K^y_i(y_i).
\]

However, we will show in the following propositions that what really matters is the property of a firm’s marginal public good costs. In the case of independent and increasing marginal costs of the public good and with further technical

\(^4\)In this case we assume that \( K^y(Y) \) is convex in \( Y \).

\(^5\)see e.g. Mendys-Kamphorst (2003)
assumption, we will show the existence of a unique market equilibrium. In contrast, if the public good’s marginal costs are dependent or constant then there exist multiple equilibria or corner solutions (Proposition 2 and 3).

**Proposition 1**

If the public good’s marginal costs are independent and increasing then there exists a unique Nash Equilibrium where the firms produce \( x^* = (x^*_1, x^*_2, ..., x^*_N) \) of the private good and \( y^* = (y^*_1, y^*_2, ..., y^*_N) \) of the public good if

\[
(3 - N) \frac{\partial p}{\partial x} + (2 - N)(x_i^* \frac{\partial^2 p}{\partial x^2}) + N(x_i^* \frac{\partial^2 p}{\partial x \partial y} + \frac{\partial p}{\partial y}) < \frac{\partial^2 K_i^x(x_i)}{\partial^2 x_i}, \forall i \in \{1, ..., N\}\]

(3)

and

\[
(N - 1)(x_i^* \frac{\partial^2 p}{\partial y \partial x} + x_i^* \frac{\partial^2 p}{\partial y^2}) + \frac{\partial p}{\partial x} + (2 - N)(x_i^* \frac{\partial^2 p}{\partial y^2}) < \frac{\partial^2 K_i^y(y_i)}{\partial^2 y_i}, \forall i \in \{1, ..., N\}.\]

(4)

**Proof:**

see Appendix

To see the intuition behind this result, we write down the profit function of a firm \( i \)

\[
\pi_i = x_i^* p(x_i, X_{-i}, y_i, Y_{-i}) - K_i^x(x_i) - K_i^y(y_i),
\]

(5)

whereby

\[
X_{-i} = \sum_{j=1, j\neq i}^{N} x_j
\]

(6)

and

\[
Y_{-i} = \sum_{j=1, j\neq i}^{N} y_j.
\]

(7)
Profit maximization, given the production of all other companies, yields the following first order conditions for firm $i$:

\[
\frac{\partial \pi_i}{\partial x_i} = p + x_i \frac{\partial p}{\partial x_i} - \frac{\partial K^x_i(x_i)}{\partial x_i} = 0 \quad (8)
\]

\[
\frac{\partial \pi_i}{\partial y_i} = x_i \frac{\partial p}{\partial y_i} - \frac{\partial K^y_i(y_i)}{\partial y_i} = 0. \quad (9)
\]

Equation (8) displays the standard condition for the optimal production level of the private good where marginal costs equal marginal revenue. Equation (9) shows that the production of the public good has only an indirect effect on the profit. By raising the quantity of the public good, the consumers’ willingness to pay increases, and therefore the firm achieves a higher price for their private goods.

Proposition 1 states that there is a unique Nash Equilibrium where the firms produce $x^* = (x_1^*, x_2^*, ..., x_N^*)$ and $y^* = (y_1^*, y_2^*, ..., y_N^*)$. To see this we proceed in three steps. First, we think about the optimal production decision with respect to the private good given the production of the public good. Second, we think about the optimal production decision with respect to the public good given the production of the private good. We will see that these decisions are always well-defined and unique. In the last step we show that starting from a point of no production we will arrive at a unique Nash Equilibrium.

First step: For the optimal production level of the private good $x_i$ for firm 1 there are only two things important: The competitors’ total production of the private good (like in the normal Cournot Game) and the total production of the public good $Y$. The last point results from the fact that for firm $i$’s private good demand it does not matter if it or another firm produces one unit of the public good. Only the total production of the public good counts. For example, in order to sell its server hardware, for IBM its only important what is the quality of Linux and not who produces it.

Furthermore, the technical assumptions ensure that the best reply function mapping with respect to the private good given the production of the public good is a contraction. This means that if you start at arbitrary vector $x^0 = (x_1^0, x_2^0, ..., x_N^0)$ and look at the best replies, you get a new vector $x^1 = (x_1^1, x_2^1, ..., x_N^1)$. If you look at the best replies to this vector and so on, this sequence converges to a unique
fixed point \( x^* = (x_1^*, x_2^*, ..., x_N^*) \).

Second step: With the knowledge that every firm produces a certain quantity of \( x_i^* \) given \( Y \), we can now look at the production decision with respect to the public good \( y_i \). To maximize a firm’s profit, the marginal revenue with respect to the public good \( y_i \) must be equal to the marginal costs (see Equation 9). Due to the fact that the public good’s marginal revenue is decreasing or constant and its marginal costs are increasing, it is ensured that for every given production level of the private good \( x_i^* = (x_1^*, x_2^*, ..., x_N^*) \) and the others’ public good production \( Y_{-i} \), there exists a unique optimal quantity \( y_i^* \) which firm \( i \) should produce. Furthermore, the second technical assumption in the Proposition 1 ensures that the best reply functions with respect to the public good given the production of the private good are once again a contraction mapping. Therefore, as above, we have a unique Nash Equilibrium \( y^* = (y_1^*, y_2^*, ..., y_N^*) \) for every production of the private good \( x \).

Third step: Now, we know that for every production of the private good \( x \) (public good \( y \)) there exists a unique Nash Equilibrium \( y^* (x^*) \). The question is, if we are able to find a unique combination of \( y^* \) and \( x^* \) which imply each other. For this, we start at a point where no firm produces anything and ask if this is a Nash Equilibrium. This is obviously not the case. Every firms has an incentive to produce at least some quantity of the private good \( x_i \) which follows immediately from Assumption (A3) and (A5). But this gives the firms an incentive to produce at least something of the public good \( y_i \) which once again gives an incentive to increase the production of the private good and so on. The system converges to a unique interior Nash Equilibrium if the private and public good’s marginal costs are increasing fast enough which is stated in the two technical assumptions of Proposition 1. In the end this leads to the unique interior Nash Equilibrium \( \{(x_1^*, y_1^*), (x_2^*, y_2^*), ..., (x_N^*, y_N^*)\} \) where no deviation up or down pays off.

In the next two Propositions we want to look at the case of constant or dependent marginal costs of the public good. We show that in these cases multiplicity of equilibria and asymmetric equilibria arise.

**Proposition 2**
There are an infinite number of equilibria, whereby in all equilibria \( x^* \) is the same and the firms’ production of the public good \( y^* \) sums up to a certain constant level \( Y^* \), if

- every firm has the same private good cost function
- the public good’s marginal costs are constant and equal for all firms or the public good cost function is dependent on the others’ public good production
- \((3-N)\frac{\partial p}{\partial x} + (2-N)(x_i \frac{\partial^2 p}{\partial x \partial y} + \frac{\partial p}{\partial y}) < \frac{\partial^2 K^x(x_i)}{\partial x^2}, \forall i \in \{1, ..., N\}\)
- \(\exists i \in \{1, ...N\} \text{s.t. } x_i \frac{\partial p(\sum_{j=1}^{N} x_j^*)}{\partial y_i} < \frac{\partial K^y}{\partial y_i} .\)

**Proof:**

see Appendix

What this case distinguishes from the first case is that given a certain total production of the public good \( Y \) the marginal production costs of the public good are equal between the firms

\[
\frac{\partial K^y}{\partial y_i} \big|_{y=Y_0} = \frac{\partial K^y}{\partial y_j} \big|_{y=Y_0} \forall i, j \in \{1, ..., N\}.
\]

(10)

If the marginal costs are constant and the same for every firm this obviously true. In the case of dependent costs, it is because the total costs for producing the public good are divided equally among the firms. This implies that every firm hast to bear \( \frac{1}{N} \) of the the costs for an additional unit of the public good.

Once again, given any public good production level \( Y \), there exists a unique Nash Equilibrium \( x^* = (x_1^*, x_2^*, ..., x_N^*) \) like in the first step with independent and increasing marginal costs. This is ensured by the first technical assumption in the Proposition 2.

What is new is that given the private good production \( x \), there does not exist a unique Nash Equilibrium \( y^* = (y_1^*, y_2^*, ..., y_N^*) \). To see this, one has to realize that extreme solutions (where every firm produces nothing of the public good or every firm produces until it reaches its capacity constraint) are ruled out by the technical
assumptions (A5), (A7) and the second assumption of Proposition 2. This implies that we have an interior solution whereby the firms’ production of the public good sums up to a certain level of $Y^*$. Proposition 2 states that all production dividing rules with $\sum_{i=1}^{N} y_i = Y^*$ are Nash Equilibria.

Why was this not the case in Proposition 1? With independent and increasing marginal costs the individual production $y_i^*$ was clearly determined through the increasing marginal production costs of the public good. These are now constant and therefore a coordination problem with an infinite number of equilibria arises. For an illustration suppose two firms which produce, in a Nash Equilibrium, the same quantities of the private good given $Y$. Assume that firm 1 thinks that firm 2 is going to produce nothing of the public good. Then firm 1 should produce until its marginal revenue is equal to its marginal costs given that the other firm produces nothing. If this is this case then it is indeed optimal for the firm 2 to produce nothing because its marginal revenue and its marginal costs are the same as for the firm 1. Generally, if firm 1 is in an optimum given the production of firm 2 then firm 2 is also in an optimum given the production of firm 1. This is what leads to the infinite number of possible equilibria and causes a coordination problem between the firms. In the rest of the paper we assume that in a case with multiplicity of equilibria the symmetric firms coordinate on the equilibrium with symmetric production levels.

In the next proposition we show the properties for an asymmetric equilibrium. For simplicity, we only address the case of two firms. A generalization to the N-firms case is straightforward.

**Proposition 3**

If

Case 1: the two firms have the same private good cost functions and different constant marginal costs for producing the public good

or if

Case 2: the two firms have different private good cost functions and constant marginal costs for producing the public good

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or if

Case 3: the two firms have different private good cost functions and dependent cost functions for producing the public good

and if the conditions

\[
\frac{\partial p}{\partial x} + 2 \ast (x_i \ast \frac{\partial^2 p}{\partial x \partial y} + \frac{\partial p}{\partial y}) < \frac{\partial^2 K^y_i}{\partial^2 x_i}, \forall i \in \{1, 2\}
\]  
\[
\left| x_i \frac{\partial^2 p}{\partial y \partial x} \right| + x_i \frac{\partial^2 p}{\partial y \partial x} + \frac{\partial p}{\partial x} < \frac{\partial^2 K^y_i}{\partial^2 y_i}
\]

are fulfilled

then the firm i which would produce more of the public good given that the other firm produces nothing, will be the only producing firm with \(y_i^*\).

**Proof:**

see Appendix

To understand this result, we first examine if the stated equilibrium is indeed a Nash Equilibrium. First realize that through the technical assumptions in the Proposition 3 it is ensured (as in the cases before) that for every \(y\) there exists a unique Nash Equilibrium \(x^*\).

Now, assume that firm 1 would produce more of the public good given that the other firm produces nothing of the public good

\[
y_1^* \mid y_2 = 0 > y_2^* \mid y_1 = 0.
\]

It is obvious that firm 1 has no incentive to deviate because this quantity maximizes its profit function given that firm 2 produces nothing. Does firm 2 have an incentive to deviate by producing any quantity of the public good? It should deviate if its marginal revenue is bigger than the marginal costs of the public good. In Case 1 of Proposition 3 both firms have the same marginal revenue with respect

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6 By the technical assumptions of Proposition 3, it is ensured that an interior solution exists.
to the public good. If Equation 13 is true then it is obvious that the marginal revenue for firm 2 is smaller than the marginal costs given $y_1^* | y_2 = 0$. This is because from $y_2^* | y_1 = 0 < y_1^* | y_2 = 0$ follows that firm 2 has higher marginal costs. Hence, firm 2 has no incentive to deviate and $\{(x_1^*, y_1^* | y_2 = 0 = y_1^*), (x_2^*, y_2^* = 0)\}$ is indeed a Nash Equilibrium.

In Cases 2 and 3 of Proposition 3 the two firms have different private good cost functions. This leads to an asymmetric quantity of the private good $x_i^*$ and therefore to different marginal revenue functions with respect to the public good $\frac{\partial R_i}{\partial Y}$. In the cases of constant and symmetric marginal costs of the public good or dependent marginal costs of the public good it is obvious that the equilibrium of Proposition 3 is indeed a Nash Equilibrium because the marginal revenue is different and the marginal costs (given a certain production of the public good) are the same. With different constant marginal costs and different marginal revenues one can see that if $y_1^* | y_2 = 0 > y_2^* | y_1 = 0$ then the production of an additional unit of the public good has no value for firm 2.\(^7\)

The uniqueness follows from the fact that if firm 2 reduces its production of the public good by one unit then firm 1 would increase its production by one unit. Therefore, the properties of the public good cost function described in Proposition 3 let $y_1$ and $y_2$ be perfect strategic substitutes. Furthermore, one can see that after the elimination of strictly dominated strategies only the described corner solution survives. For an illustration look at firm 2 with $y_2^* | y_1 = 0 < y_1^* | y_2 = 0$. It knows that firm 1 always produces at least the difference between $y_1^* | y_2 = 0 - y_2^* | y_1 = 0 = y_1^{min}$. But with this knowledge firm 2 should produce at maximum $y_2^* | y_1 = y_2^{min} < y_2^* | y_1 = 0$. This implies a new production of firm 1 $y_1^* | y_2 = 0 - y_2^* | y_1 = y_1^{min} = y_1^{min'} > y_1^{min}$ which implies once again a new maximum production of firm 2 and so on. One easily sees that this ends at the described corner solution of Proposition 3.

### 4 Change of a Incumbent’s Profit and the Production Adjustment

In this section we want to study the general case. In the next section we will apply it with specific functions. We look on how an incumbent’s profit and its

\(^7\)We show in the Appendix that under these circumstances its marginal revenue is smaller than its marginal costs given the production $y_1^* | y_2 = 0$. 

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production decisions change if the others’ production of the public and private good changes for what ever reason. We therefore assume that the firms are in a Nash Equilibrium before the exogenous shock.\footnote{We restrict our attention to a firm which production decision is not influenced by its capacity constraints before and after its adjustment. Otherwise one has to check if these constraint are fulfilled which leads to no new insights but only makes everything more complicated.}

**Change of the Profit**

Suppose that firm $i$ faces a change in the others’ total production of the private good $X_{-i}$ and the public good $Y_{-i}$. This has pecuniary externalities on firm $i$’s profit. An increase in the private good production $X_{-i}$ has a negative pecuniary externality because it decreases the price for the private good in the market. An increase in the public good production $Y_{-i}$ has a positive externality because it increases, due to the complementarity in the demand function, the price of the private good. These insights lead to Proposition 4:

**Proposition 4**

If firm $i$ faces a change in the production of the private good $X_{-i}$ and in the production of the public good $Y_{-i}$ then the price of the private good in the market can change. If the change in $X_{-i}$ and $Y_{-i}$ is small and the new price is higher (lower) then the firm’s profits increase (decrease). If the price does not change then the incumbent’s profit is not affected.

**Proof:**

see Appendix

One may wonder if this analysis is really so simple. Indeed, a change in the production of the other firms has two effects on the profit. First, it changes the price for the private good and therefore influences the profit directly. Second, the firm reacts to the new production level of the others and adjust its own production levels which can also influence the firm’s profit. Nevertheless, we know from the Envelope Theorem that given we start in an optimum and as long as the changes
are small this second order effect is close to zero and could be neglected.

**Adjustment of the Production**

After considering the changes in the profit, we now want to look at the adjustment of the production decision.

This is not so easy as it looks like because one has to keep in mind cross effects. To see this, the easiest way is to distinguish between direct and indirect effects. Taking the total derivative of the firm’s first order conditions, one gets Equations 14 and 15.

Total derivative of \( \frac{\partial \pi_i}{\partial x_i} \):

\[
\left[ 2 \frac{\partial p}{\partial x_i} + x_i \frac{\partial^2 p}{\partial x_i^2} - \frac{\partial^2 K(x_i)}{\partial x_i^2} \right] dx_i + \left[ \frac{\partial p}{\partial y_i} + x_i \frac{\partial^2 p}{\partial x_i \partial y_i} \right] dy_i
\]

\[
\left( \text{indirect effect} \right)
\]

\[
\left( \text{direct effects} \right)
\]

\[
+ \left[ \frac{\partial p}{\partial X_{-i}} + x_i \frac{\partial^2 p}{\partial x_i \partial X_{-i}} \right] dX_{-i} + \left[ \frac{\partial p}{\partial Y_{-i}} + x_i \frac{\partial^2 p}{\partial x_i \partial Y_{-i}} \right] dY_{-i} = 0
\]

Total derivative of \( \frac{\partial \pi_i}{\partial y_i} \):

\[
\left[ \frac{\partial p}{\partial y_i} + x_i \frac{\partial^2 p}{\partial x_i \partial y_i} \right] dx_i + \left[ x_i \frac{\partial^2 p}{\partial y_i^2} - \frac{\partial^2 K_i(y_i)}{\partial y_i^2} \right] dy_i
\]

\[
\left( \text{indirect effect} \right)
\]

\[
\left( \text{direct effects} \right)
\]

\[
+ \left[ x_i \frac{\partial^2 p}{\partial y_i \partial X_{-i}} \right] dX_{-i} + \left[ x_i \frac{\partial^2 p}{\partial y_i \partial Y_{-i}} \right] dY_{-i} = 0
\]

Equation (14) and (15) show that we can decompose the overall effect into direct and indirect effects in order to understand the complex adjustment process:

1. The direct effect influences \( x_i \) (\( y_i \)) through a change in \( X_{-i} \) or \( Y_{-i} \) without depending on a change in the corresponding complement \( y_i \) (\( x_i \)).
2. The indirect effect influences the optimal level of \( x_i \) (\( y_i \)) through a change in
the corresponding complement $y_i (x_i)$.

For an illustration suppose that IBM faces an exogenous increase in the quality of the Linux. On the one hand, this may increase the consumers’ valuation of IBM’s server and therefore gives IBM an incentive to increase its production of the hardware (= a direct effect of $Y_{-i}$ on $x_i$). On the other hand, the additional supply of the public good might decrease the firm’s production of the public good because keeping the old production level would lead to a higher overall amount of the public good (= a direct effect of $Y_{-i}$ on $y_i$). Furthermore, there are indirect effects. If IBM produces more server this may change the incentive to invest in the public good Linux (=indirect effect of $x_i$ on $y_i$). Additionally, if IBM changes its investment in Linux, the quality of the operation system changes and this will have an effect on the incentives to produce the private good server (= indirect effect of $y_i$ on $x_i$).

In the next section we want to use these insights to analyze the effect of government intervention and the consequences of competition.

5 The Analysis of Government Intervention and the Effects of Market Entry

In this section we first want to analyze what happens if the government engages in the production of the public good. A government intervention is not a hypothetical question as one can see from the U.S. Government support of Linux. In a second step, we want to consider the case of market entry and determine the consequences on the incumbents’ production and profits.

Government Intervention

If the government increases its contribution to the public good, the effect on an incumbent’s profit is straightforward. The additional supply of the public good increases the price of the private good and therefore, as stated in Proposition 4, the profits increase.

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9 Evans and Reddy (2002) p.2
What is not so obvious is the adjustment of the incumbent’s production. The standard public good literature states that if the government increases its investment in a public good, like the social security system, then the agents usually decrease their donations to the public good.\textsuperscript{10} In our context, we will show that this does not have to be true and that even the opposite may happen.\textsuperscript{11}

At first, we want to look at the case of a total crowding out meaning that the incumbents would reduce their production of the public good by 1:1 if the government increases its investment in the public good. Furthermore, this leads to the fact that the incumbents do not adjust their production of the private good.

**Proposition 5**

*An increase in the government investment in the public good leads to a 1:1 reduction of the firms’ production of the public good without changing the production level of the private good if*

- the marginal revenue of the public good is decreasing ($\frac{\partial^2 R_i}{\partial y^2} < 0$)
- the marginal production costs of the public good are constant ($\frac{\partial^2 K_y^i}{\partial y^i} = 0$).

**Proof:**

From Proposition 2 we know that with constant marginal costs the total public good production have to sum up to $Y^*$. In addition we know that under these circumstances the public good exhibits the properties of a perfect substitute.

Q.E.D.

Due to the fact that the marginal costs of the public good are constant and the marginal revenue of the public good is decreasing, there is an intersection where the marginal revenue is equal to the marginal cost. This point determines the optimal production level of the public good. If now the state increases the production of the public good, holding everything else constant, then the marginal costs

\textsuperscript{10}see e.g. Bergstrom, Blume and Varian (1985)

\textsuperscript{11}Because one can consider several different cases we restrict our attention to only two examples which highlight the possible outcomes.
are higher than the marginal revenue because the marginal revenue is decreasing. Therefore, the firms get an incentive to decrease their production of the public good. They should decrease their production until the marginal costs are again equal marginal revenue. And this is achieved by a 1:1 reduction. In the end, the overall quantity of the public good does not change and this leads to no adjustment of the private good production. Form a welfare point-of-view, the consumers do not benefit form the government intervention. Only the incumbent gets better off. They have to produce a smaller amount of the public good by getting the same price for the private good. An intervention under these circumstances is like transferring money from the government to the firm without changing anything else.

The analysis completely changes if we assume for instance that the marginal revenue of the public good is constant with respect to the public good, the marginal costs of the public are increasing and with the technically assumption that the cross derivative of the price does not dominate.

**Proposition 6**

An increase in the government’s public good investment leads to an increase in the production of the private and the public good if

- the marginal revenue of the private good is increasing with respect to the public good ($\frac{\partial^2 R_i}{\partial x_i \partial y} > 0$)
- the marginal revenue of the public good is constant ($\frac{\partial^2 R_i}{\partial y^2} = 0$)
- the marginal production costs of the public good are increasing ($\frac{\partial^2 K_y}{\partial y_i^2} > 0$)
- the cross derivative of the price does not dominate the positive effect of the public good on the price ($\frac{\partial p}{\partial y} > -N * \frac{\partial p}{\partial y \partial x}$)

**Proof:**

see Appendix
In this case the public good’s marginal costs are increasing and the marginal revenue of the public good is constant. This leads to an intersection where the marginal costs are equal to the marginal benefit, determining the optimal production level of the public good given the production of the private good. If now the state increases its contribution to the public good then this does not change the marginal revenue with respect to the public good. Therefore, the incumbents have no direct incentive to adjust their production of the public good. As the total quantity of the public good increases the incumbents have an incentive to increase their production of the private good (=direct effect) because the private good’s price raises. This increase in the total private good’s production has a feedback-effect (= indirect effect) on the incentives to produce the public good because the marginal revenue of public good is increasing in the production of the private good. The technical condition in the proposition states that the direct effect of the others’ change in the private good production does not dominate the indirect effect through the firm’s own production change. This is for example fulfilled if the cross derivative of the price is positive or zero. In the end, this leads to a higher production level of the incumbent and increases the overall quantity of the private and the public good.

We have seen that the effect of a government intervention is not straightforward. One has to look closely on the properties of the market to determine if it is a strategic complement or substitute to the firm’s investment decisions.

**Effects of Market Entry**

Now, we want to consider the effects of market entry. So far, we have assumed that the incumbents face a given exogenous shock in the quality of the public good. With market entry the size of this shock is determined endogenously and further more the incumbents face a change in the overall private good production through the entrant. To be able to make statements about the new Nash-Equilibrium we restrict ourselves to a setup with symmetric firms which have quadratic cost functions and consumers that have a linear-demand function. Nevertheless, we are able to point out the crucial issues.

We therefore assume an inverse demand function which is linear in \( x \) and \( y \)
whereby the impact of $x$ and $y$ on the price is determined by the parameters $b$ and $c$.

$$p = A - bX + cY$$  \hfill (16)

Even if such a linear demand function is a special case it could be seen as an approximation of every possible demand function because the parameters $b$ and $c$ allow for all kind of slopes and so one can use such a linear demand function as long as one only considers a small pieces of the function.

Consider $N$ symmetric firms and assume for all firms quadratic cost functions.

$$K^x_i(x_i) = dx_i^2$$  \hfill (17)

$$K^y_i(y_i) = fy_i^2$$  \hfill (18)

Hereby, the parameters $d$ and $f$ represent the weight of the cost functions with which they influence the firm’s profit.

To ensure an interior symmetric solution we introduce the Lemma 1 and assume that the mentioned conditions are always fulfilled.

**Lemma 1**

There exists a unique interior solution $\{x^*, y^*\}$ where every firm produces the same quantity $x^*_i$ and $y^*_i$ if the following conditions are fulfilled:

- If $f \geq \frac{c^2}{2b}$ then $N < \frac{3b+2d-c^2}{b-c^2f}$
- If $f < \frac{c^2}{2b}$ then $N < \frac{b+2d}{b+c^2f}$

**Proof:**

see Appendix

The conditions in the Lemma 1 are without loss of generality because we can allow for all values of $N$ through choosing a high enough $d$. Intuitively, the Lemma
1 ensures that the firms’ best reply functions are a contraction mapping. This is true if the costs for producing the private good have enough weight.

To find the optimal production values we write down the profit function of a firm $i$, whereby we denote by $x_j$ and $y_j$ the production of firm $j \in \{1, ..., N\} \setminus i$:

$$\pi_i = x_i \ast (A - bx_i - (N - 1)bx_j + cy_i + (N - 1)cy_j) - dx_i^2 - fy_i^2.$$  \hfill (19)

The first-order conditions are

$$\frac{\partial \pi_i}{\partial x_i} = A - 2bx_i - (N - 1)bx_j + cy_i + (N - 1)cy_j - 2dx_i = 0$$  \hfill (20)

and

$$\frac{\partial \pi_i}{\partial y_i} = cx_i - 2fy_i = 0.$$  \hfill (21)

Solving (21) for $y_i^*$ yields to

$$y_i^* = \frac{c}{2f} x_i.$$  \hfill (22)

Equation (22) shows that $y_i^*$ depends only on the firm’s production of the private good and on the weight of the public good’s cost function $f$ and the impact of $y$ on the price. Therefore, the optimal level of $y_i^*$ is independent of the production decision of the other firms and changes only if $x_i^*$ changes. We summarize this observation in the following Lemma.

**Lemma 2**

Each firm produces the private good $x_i^*$ and the public good $y_i^*$ in the same ratio which is determined by the impact of the public good on the price relative to the weight of the public good cost function on the firm’s profit:

$$y_i^* = \frac{c}{2f} x_i.$$  

**Proof:**

See Equation 22.
The intuition for this is as follows: For a firm, the production of a public good has only the effect of increasing the price of the private good. In our case the effect of $y_i$ on $p$ is always constant and equal to $c$. The marginal revenue of $y_i$ is therefore $cx_i$. The marginal costs are $2fy_i$. In the optimum, the marginal revenue must equal the marginal costs. Therefore, we can see that the relationship between $x_i$ and $y_i$ is linear and the constant slope depends only on the weights of the public good’s production cost $f$ and the impact of $y$ on the price $p$. \footnote{In general this is of course not true. The linear demand function has the property that all second order conditions are zero, so the cross derivative too. Therefore, one has not to take into account that the marginal revenue of the public good is dependent of the others private and public good production.}

Next, we want to determine the optimal quantity of firm $i$’s private good. Solving (20) for $x_i^*$ and using (22) and symmetry yields to

$$x_i^* = \frac{A}{b(1 + N) - \frac{f^2}{2f} N + 2d}.$$  \hfill (23)

After deriving the firms’ optimal production decisions we want to look how these change due to market entry. Therefore, suppose that an additional firm enters the market that can use the same production technology as the incumbents. This means that the new firm is going to produce the private and the public good which leads to two effects: On the one hand competition increases because of the additional production of the private good which may lower the incentives to produce $x_i$. On the other hand the entrant’s production of the public good raises the consumers’ valuation of the private good and therefore gives an incentive to increase the production of the private good. Hence, it is not straight forward in which direction a incumbent is going to adjust its production of the private good.
Proposition 7

If the number of competing firms \( N \) increases then

- each incumbent reduces its production of \( x_i^* \) and \( y_i^* \) if the production costs of the public good are high relative to the price impact of the public good in terms of the private good’s weight in the demand function \((f > \frac{c^2}{2b})\)

- each incumbent does not change its production of \( x_i^* \) and \( y_i^* \) if the production costs of the public good have a medium value relative to the price impact of the public good in terms of the private good’s weight in the demand function \((f = \frac{c^2}{2b})\)

- each incumbent increases its production of \( x_i^* \) and \( y_i^* \) if the production costs of the public good are low relative to the price impact of the public good in terms of the private good’s weight in the demand function \((f < \frac{c^2}{2b})\).

Proof:

To proof Proposition 7 we take the first derivative of \( x_i^* \) with respect to \( N \).

\[
\frac{\partial x_i^*}{\partial N} = -\frac{A}{\left(b(1+N) - \frac{1}{2} c^2 N + 2d\right)^2} \left(b - \frac{1}{2} \frac{c^2}{f}\right)
\]  

(24)

We can see that the sign is negative if \( f > \frac{c^2}{2b} \), is zero if \( \frac{c^2}{2b} \) and is positive if \( f < \frac{c^2}{2b} \).

Furthermore, we know through Lemma 2 that \( x_i^* \) and \( y_i^* \) behave direct proportional to each other.

Q.E.D.

Proposition 7 states the somehow surprising result that the effect on \( x_i^* \) depends only on the weight of the public good’s cost function relative to the impact of the private good and of the public good on the price. This issue becomes clear if we consider the effects of a new firm on the incumbents. In the equilibrium, the new firm produces the same quantity of the public and of the private good as every
incumbent after the adjustment process. Due to the chosen demand function, \( x \) and \( y \) influence the price \( p \) always with a constant weight. Therefore, if in the new equilibrium the additional competitor has with its overall production of the private good the same impact on the price as with its overall production of the public good then the firm does not influence the behavior of the incumbents. This is the case because the entrant does not alter the price and therefore in some way the incumbents do not even notice the market entry.

If the weight of the public good’s cost function is small enough \( (f < \frac{c^2}{2b}) \) then in an equilibrium the entrant produces enough of the public good to overcompensate its negative pecuniary externality through the private good production. This leads to an increase in the price and therefore gives the incumbents an incentive to increase the private good production and, as a direct consequence, also the public good production.

If the weight of the public good’s cost function is high \( (f > \frac{c^2}{2b}) \) then the opposite is true.

Next, we want to look at the variation of the overall production of the private and public good. By Proposition 7 we know that under some circumstances every firm decreases its production of the private and the public good. Therefore, it is questionable if the additional production of the entrant can compensate this decline of the incumbents’ production.

**Proposition 8**

*Market entry unambiguously increases the overall production of the private good \( X \) and the public good \( Y \).*

**Proof:**

see Appendix

Therefore, we see that the production of the additional firm is always big enough to compensate for the loss of production by the incumbents. This implies an interesting consequence:
The outcome of a normal Cournot-Game with the number of firms progressing towards infinity is equivalent to the outcome of a game with perfect competition where the firms behave as price takers.

In our setup the production of the public good should break down if the firms behave as price takers because the motivation for the production of the public good is the change of the private good’s price. Hence, in our model the market equilibrium of a Cournot-Game with an infinite number of firms is no longer equivalent to the market equilibrium where the firms behave as price taker because the equilibria are different.

With the knowledge about the change in the total production of the private and the public good we can determine the variation of the price.

**Proposition 9**

*If the number of competing firms $N$ increases then*

- the price $p$ decreases if the production costs of the public good are high relative to the price impact of the public good in terms of the private good’s weight in the demand function ($f > \frac{c^2}{2b}$)
- the price $p$ does not change if the production costs of the public good have a medium value relative to the price impact of the public good in terms of the private good’s weight in the demand function ($f = \frac{c^2}{2b}$)
- the price $p$ increases if the production costs of the public good are low relative to the price impact of the public good in terms of the private good’s weight in the demand function ($f < \frac{c^2}{2b}$).

**Proof:**

see Appendix

The result gets intuitively clear if one takes into account the total change in the overall production. We know by Lemma 2 that the ratio between the public and private good is always constant. Furthermore, we know by Proposition 8 that the overall production is increasing in $N$. This implies that the absolute difference
between the overall production of the private and public good is increasing with more active firms. If \( f \) is small enough this implies that the public good production is so high relative to the private good production that an overall increase in the production (with keeping the ratio between the private and public good production constant) will increase the absolute values in such a way that the price is increasing. Only in the case where every firm produces so much of the private good relative to the public good that it does not alter the price an increase in the overall production will have no effect on the price.

After analyzing the changes of the price and the adjustment of the production, we now want to look at the firms’ profits and the social surplus.

In a “normal” Cournot-Game, i.e. with only one homogeneous private good, the incumbents dislike market entry since the entrant has a negative pecuniary externality through the additional supply of the private good on the active firms. In contrast, in our setup the entrant has also a positive pecuniary externality since the entrant contributes to the public good.

**Proposition 10**

*If the number of competing firms \( N \) increases then*

- *the profit of the incumbents decreases if the production costs of the public good are high relative to the price impact of the public good in terms of the private good’s weight in the demand function (\( f > \frac{c^2}{2b} \))*

- *the profit of the incumbents does not change if the production costs of the public good have a medium value relative to the price impact of the public good in terms of the private good’s weight in the demand function (\( f = \frac{c^2}{2b} \))*

- *the profit of the incumbents increases if if the production costs of the public good are low relative to the price impact of the public good in terms of the private good’s weight in the demand function (\( f < \frac{c^2}{2b} \)).*

**Proof:**

see Appendix
Through Proposition 7 we know that with market entry and a high weight of the public good’s cost function \( f > \frac{c_2^2}{2b} \) every firm produces less of both goods and that the price in the market decreases. Hence, it is straightforward that under these circumstances the profit of the incumbents decrease, which is in line with the usual effect of tougher competition. If \( f < \frac{c_2^2}{2b} \), we get the surprising result that the incumbents prefer more competition. This is due to the fact that a symmetric competitor does not only produce the private good which has a negative pecuniary external effect on the incumbents but also produces some quantity of the public good and therefore has a positive pecuniary external effect on the incumbents. If the weight of the public good’s production cost is small then the positive external effect dominates the negative effect.

One might think that the social surplus reacts ambiguously to a market entry because sometimes the firms get higher profits and the price for the private good rises. We are able to show in Proposition 11 that this is not the case.

**Proposition 11**

*If the number of competing firms \( N \) increases then the social surplus always increases.*

**Proof:**

see Appendix

If we just look at the consumer surplus, we see that it is increasing with the number of active firms. This can be explained by two effects. First, through market entry the production of the public good increases. This leads to a higher private good valuation which has a positive effect on the consumer surplus. Second, the market entry leads to a fiercer competition in the proprietary sector which increases the total production and has once again a positive effect on the consumer surplus. So, even if the market entry leads to higher prices we get a higher consumer surplus. In our setup the relevant issue is the total production of both goods. Since the firms cannot make any price discrimination a higher production level of the goods leads to a higher consumer surplus.
Proposition 10 shows that with a low weight of the public good’s production cost market entry increases the profits of the firms. In this case it is obvious that the total surplus also increases. Nevertheless, if the firms’ profit decrease through the market entry the gain in the consumer surplus compensates this effect and results in a higher total surplus. Therefore, a social planner always prefers market entry.

6 Conclusion

In this paper we have studied the incentive to contribute to a non-excludable public good if it is complementary to a private good. We have shown that an intervention of the government in the public good production can lead to ambiguous results. Firms may decrease their production and a crowding out may occur. In this case the government’s contribution is a strategic substitute to the firms’ public good production. But it also possible that the firms increase their production level of the private and of the public good which implies that the activity of the government is a strategic complement.

This fact leads to a policy implication: If a government thinks about supporting Open Source Software through increasing the quality, the fear that at least a partially crowding out occurs can be reasonless. Exact the opposite can be true: The government’s activity motivates the firms to increase their contribution to public good combined with an increase in their private good production.

Furthermore, we have looked at the case of market entry. We showed that an entrant has positive and negative pecuniary externalities on the incumbents. In the end there can be the surprising effect that under some circumstances an incumbent likes market entry.

This can throw light on the incentives to license a proprietary product to competitors and therefore to allow for competition or even to welcome competitors in a market.

Usually, it is argued that licensing allows a firm to commit to higher quality levels or to a higher network through increasing competition. This leads to an
overall higher demand and offsets the loss in market power.\textsuperscript{13} An other argument, which comes up with our paper, might be that the firm use licensing as a device to allow for market entry. The additional firm will not only produce the private good but also supply public goods like e.g. cooperative advertising which compensates for the competition in the proprietary sector and leads to an overall higher profit of the licenser.

There is this famous example of Apple Computer that responded to the market entry of IBM with full-page newspaper advertisements headed "Welcome IBM. Seriously" (see Appendix). Apple claims in these advertisements that there will be a huge market if the people understand the value of a PC. And it looks like as Apple thinks that IBM will help Apple to convince people: "We look forward to responsible competition in the massive effort to distribute this American technology to the world". This is in line with our argumentation if one sees the "massive effort" as a public good because it will increase the demand of both companies.

\textsuperscript{13}See e.g. Shepard (1987) and Economides (1997).
Appendix

Proof of Proposition 1:

We proceed in two steps. First we prove existence and afterwards uniqueness.

1. Existence:
   From the Assumptions A1 follows that the strategy spaces $S_i = X_i \times Y_i$ with $X_i \in [0; X_i]$ and $Y_i \in [0; Y_i]$ are nonempty compact convex subsets of $R^2$. Furthermore by Assumptions A3, A4, A6 and A7 the profit functions $\pi_i$ are continuous in $s$ and quasi-concave in $s_i$. Therefore, it follows immediately that there exists a pure-strategy Nash equilibrium (Debreu, 1952).

2. Uniqueness
   To show uniqueness we apply the contraction mapping approach. Due to Bertsekas (1999) it is sufficient to show that the Hessian of the profit functions fulfills the "diagonal dominance" condition.

$$H = \begin{vmatrix}
\frac{\partial^2 \pi_1}{\partial x_1 \partial x_1} & \frac{\partial^2 \pi_1}{\partial x_1 \partial y_1} & \frac{\partial^2 \pi_1}{\partial x_1 \partial y_2} & \cdots & \frac{\partial^2 \pi_1}{\partial x_1 \partial y_n} \\
\frac{\partial^2 \pi_2}{\partial x_1 \partial x_1} & \frac{\partial^2 \pi_2}{\partial x_1 \partial y_1} & \frac{\partial^2 \pi_2}{\partial x_1 \partial y_2} & \cdots & \frac{\partial^2 \pi_2}{\partial x_1 \partial y_n} \\
\frac{\partial^2 \pi_1}{\partial x_2 \partial x_1} & \frac{\partial^2 \pi_1}{\partial x_2 \partial y_1} & \frac{\partial^2 \pi_1}{\partial x_2 \partial y_2} & \cdots & \frac{\partial^2 \pi_1}{\partial x_2 \partial y_n} \\
\frac{\partial^2 \pi_2}{\partial x_2 \partial x_1} & \frac{\partial^2 \pi_2}{\partial x_2 \partial y_1} & \frac{\partial^2 \pi_2}{\partial x_2 \partial y_2} & \cdots & \frac{\partial^2 \pi_2}{\partial x_2 \partial y_n} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{\partial^2 \pi_n}{\partial x_1 \partial x_1} & \frac{\partial^2 \pi_n}{\partial x_1 \partial y_1} & \frac{\partial^2 \pi_n}{\partial x_1 \partial y_2} & \cdots & \frac{\partial^2 \pi_n}{\partial x_1 \partial y_n} \\
\frac{\partial^2 \pi_n}{\partial x_2 \partial x_1} & \frac{\partial^2 \pi_n}{\partial x_2 \partial y_1} & \frac{\partial^2 \pi_n}{\partial x_2 \partial y_2} & \cdots & \frac{\partial^2 \pi_n}{\partial x_2 \partial y_n} \\
\frac{\partial^2 \pi_n}{\partial x_n \partial x_1} & \frac{\partial^2 \pi_n}{\partial x_n \partial y_1} & \frac{\partial^2 \pi_n}{\partial x_n \partial y_2} & \cdots & \frac{\partial^2 \pi_n}{\partial x_n \partial y_n} \\
\frac{\partial^2 \pi_n}{\partial y_1 \partial x_1} & \frac{\partial^2 \pi_n}{\partial y_1 \partial y_1} & \frac{\partial^2 \pi_n}{\partial y_1 \partial y_2} & \cdots & \frac{\partial^2 \pi_n}{\partial y_1 \partial y_n} \\
\frac{\partial^2 \pi_n}{\partial y_2 \partial x_1} & \frac{\partial^2 \pi_n}{\partial y_2 \partial y_1} & \frac{\partial^2 \pi_n}{\partial y_2 \partial y_2} & \cdots & \frac{\partial^2 \pi_n}{\partial y_2 \partial y_n} \\
\frac{\partial^2 \pi_n}{\partial y_n \partial x_1} & \frac{\partial^2 \pi_n}{\partial y_n \partial y_1} & \frac{\partial^2 \pi_n}{\partial y_n \partial y_2} & \cdots & \frac{\partial^2 \pi_n}{\partial y_n \partial y_n}
\end{vmatrix}$$

Therefore, the diagonal of the Hessian dominates the off-diagonal entries if

$$\sum_{j=1, j\neq i}^N |\frac{\partial^2 \pi_i}{\partial x_i \partial x_j}| + \sum_{j=1}^N |\frac{\partial^2 \pi_i}{\partial x_i \partial y_j}| < |\frac{\partial^2 \pi_i}{\partial x_i \partial x_i}|, \forall i \quad (26)$$

and

$$\sum_{j=1}^N |\frac{\partial^2 \pi_i}{\partial y_i \partial x_j}| + \sum_{j=1, j\neq i}^N |\frac{\partial^2 \pi_i}{\partial y_i \partial y_j}| < |\frac{\partial^2 \pi_i}{\partial y_i \partial y_i}|, \forall i \quad (27)$$
are fulfilled.

Calculating the derivatives and plugging into Equation 26 leads to

\[
\sum_{j=1,i\neq j}^{N} |x_i^* \frac{\partial^2 p}{\partial x_i \partial x_j}| + \sum_{j=1}^{N} |x_i^* \frac{\partial^2 p}{\partial x_i \partial y_j} + \frac{\partial p}{\partial y_j}| < |x_i^* \frac{\partial^2 p}{\partial x_i \partial x_i} + 2 \frac{\partial p}{\partial x_i} - \frac{\partial^2 K^x_i(x_i)}{\partial x_i \partial x_i}|, \forall i
\]

\[ (28) \]

\[
(N-1)|x_i^* \frac{\partial^2 p}{\partial x_i \partial x_j}| + N|x_i^* \frac{\partial^2 p}{\partial x_i \partial y_j} + \frac{\partial p}{\partial y_j}| < |x_i^* \frac{\partial^2 p}{\partial x_i \partial x_i} + 2 \frac{\partial p}{\partial x_i} - \frac{\partial^2 K^x_i(x_i)}{\partial x_i \partial x_i}|, \forall i
\]

\[ (29) \]

\[
(1-N)(x_i^* \frac{\partial^2 p}{\partial x_i \partial x_i} + \frac{\partial p}{\partial x_i}) + N(x_i^* \frac{\partial^2 p}{\partial x_i \partial y_i} + \frac{\partial p}{\partial y_i}) < -(x_i^* \frac{\partial^2 p}{\partial x_i \partial x_i} + 2 \frac{\partial p}{\partial x_i} - \frac{\partial^2 K^x_i(x_i)}{\partial x_i \partial x_i}), \forall i
\]

\[ (30) \]

\[
(2-N)(x_i^* \frac{\partial^2 p}{\partial x_i \partial x_i} + \frac{\partial p}{\partial x_i}) + N(x_i^* \frac{\partial^2 p}{\partial x_i \partial y_i} + \frac{\partial p}{\partial y_i}) < -(\frac{\partial p}{\partial x_i}) + \frac{\partial^2 K^x_i(x_i)}{\partial x_i \partial x_i}, \forall i
\]

\[ (31) \]

\[
(3-N) \frac{\partial p}{\partial x} + (2-N)(x_i^* \frac{\partial^2 p}{\partial x_i \partial x_i}) + N(x_i^* \frac{\partial^2 p}{\partial x_i \partial y_i} + \frac{\partial p}{\partial y_i}) < \frac{\partial^2 K^x_i(x_i)}{\partial x_i \partial x_i}, \forall i
\]

\[ (32) \]

Calculating the derivatives and plugging into Equation 27 leads to

\[
\sum_{j=1,i\neq j}^{N} |x_i^* \frac{\partial^2 p}{\partial y_i \partial x_j}| + |x_i^* \frac{\partial^2 p}{\partial y_i \partial y_i} + \frac{\partial p}{\partial y_i}| + \sum_{j=1,i\neq j}^{N} |x_k^* \frac{\partial^2 p}{\partial y_k \partial y_i}| < |x_i^* \frac{\partial^2 p}{\partial y_i \partial y_i} - \frac{\partial^2 K^y_i(y_i)}{\partial y_i \partial y_i}|, \forall i
\]

\[ (33) \]

\[
(N-1)|x_i^* \frac{\partial^2 p}{\partial y_i \partial x_j}| + |x_i^* \frac{\partial^2 p}{\partial y_i \partial x_i} + \frac{\partial p}{\partial x_i}| + (N-1)|x_i^* \frac{\partial^2 p}{\partial y_i \partial y_j}| < |x_i^* \frac{\partial^2 p}{\partial y_i \partial y_i} - \frac{\partial^2 K^y_i(y_i)}{\partial y_i \partial y_i}|, \forall i
\]

\[ (34) \]

\[
(N-1)|x_i^* \frac{\partial^2 p}{\partial y_i \partial x_i} + x_i^* \frac{\partial^2 p}{\partial y_i \partial x_i} + \frac{\partial p}{\partial x_i} + (1-N)(x_i^* \frac{\partial^2 p}{\partial y_i \partial y_j}) < -(x_i^* \frac{\partial^2 p}{\partial y_i \partial y_i}) + \frac{\partial^2 K^y_i(y_i)}{\partial y_i \partial y_i}, \forall i
\]

\[ (35) \]
\[(N - 1)|x_k^* \frac{\partial^2 p}{\partial y_i \partial x_j}| + x_i^* \frac{\partial^2 p}{\partial y_i \partial x_j} + \frac{\partial p}{\partial x_i} + (2 - N)(x_i^* \frac{\partial^2 p}{\partial y_i \partial y_j}) < \frac{\partial^2 K^y_i(y_i)}{\partial y_i \partial y_i}, \forall i \quad (36)\]

\[(N - 1)|x_i^* \frac{\partial^2 p}{\partial y \partial x}| + x_i^* \frac{\partial^2 p}{\partial y \partial x} + \frac{\partial p}{\partial x} + (2 - N)(x_i^* \frac{\partial^2 p}{\partial y \partial y}) < \frac{\partial^2 K^y_i(y_i)}{\partial^2 y_i}, \forall i \quad (37)\]

Q.E.D.

Proof of Proposition 2:

We proceed in two steps. First we prove existence and show afterwards the multiplicity of equilibria.

1. Existence

See first part of the proof of Proposition 1.

2. Multiplicity

We proceed in 5 steps:

Step 1:
Due to the first technical assumption in the Proposition 2 we have for every \(y\) a contraction mapping of the best reply functions with respect to the private good. This leads to a unique Nash Equilibrium \(x^* = (x_1^*, x_2^*, ..., x_N^*)\) for every \(y\).

Step 2:
By Assumptions (A4),(A5) and the second technical assumption in the Proposition 2 the Nash Equilibrium cannot be

\[s^* = \{(x_1^*, y_1^* = 0), (x_2^*, y_2^* = 0), ..., (x_j^*, y_j^* = 0), (x_k^*, y_k^* = 0), ..., (x_N^*, y_N^* = 0)\}\]

or

\[s^* = \{(x_1^*, \overline{y}_1), (x_2^*, \overline{y}_2), ..., (x_j^*, \overline{y}_j), (x_k^*, \overline{y}_k), ..., (x_N^*, \overline{y}_N)\}\].
Step 3:
By the proof of existence we know that there exists at least one Nash Equilibrium. Therefore, we can assume that
\[ s^* = (x_1^*, y_1^*), (x_2^*, y_2^*), \ldots, (x_j^*, y_j^*), (x_k^*, y_k^*), \ldots, (x_N^*, y_N^*) \]
is a Nash Equilibrium.

Step 4:
By Step 2 it is possible to find a \( y_j \neq y_k \) or a \( y_j = y_k \) with \( y_j \in [0, Y_j] \). Furthermore, it is possible to find a \( \mu \in R \) s.t. \( y_j + \mu = y_j' \in [0, Y_j] \land y_k - \mu = y_k' \in [0, Y_k] \).

\( s^* \) implies that the FOCs of every firm \( i \in \{1, \ldots, N\} \) must be fulfilled at the values of \( s^* \):
\[
\frac{\partial \pi_i}{\partial x_i} = \frac{\partial R_i}{\partial x_i} - \frac{\partial K_i^x}{\partial x_i} = 0
\]
\[
\frac{\partial \pi_i}{\partial y_i} = \frac{\partial R_i}{\partial y_i} - \frac{\partial K_i^y}{\partial y_i} = 0
\]

If this is the case then these first order conditions are also fulfilled with the values \( s' = ((x_1^*, y_1^*), (x_2^*, y_2^*), \ldots, (x_j^*, y_j^*), (x_k^*, y_k^*), \ldots, (x_N^*, y_N^*)) \).

To see this, realize that \( Y = \sum_{i=1}^{k} y_i \) does not change. By
\[
\frac{\partial K_i^y}{\partial y_i} \bigg|_Y \equiv \frac{\partial K_j^y}{\partial y_j} \bigg|_{Y \forall i, j \in \{1, \ldots, N\}}
\]
the marginal costs of all firms remain the same. Furthermore, the firms' marginal revenues do not change. This leads to the conclusion that all first order conditions are fulfilled at \( s' \).

If all first order conditions are fulfilled then \( s' \) is a Nash Equilibrium.

Step 5:
Now one can go back to Step 4 and by using \( s' \) instead of \( s^* \) with a different \( \mu \) this leads to a new Nash Equilibrium.

Through repeating Step 4 and 5 it is obvious that there exist an infinite number of equilibria.

Q.E.D.
Proof of Proposition 3:

We will first prove that the described equilibrium is indeed a Nash Equilibrium and afterwards show that this is the only Nash Equilibrium by ruling out all other possible equilibria.

1. Is \( \{ (x_1^*, y_1^{|y_2=0}) , (x_2^*, y_2^{|y_1=0}) \} \) with \( y_1^{|y_2=0} > y_2^{|y_1=0} \) a Nash Equilibrium?

The FOCs of the firms with respect to the public good are

\[
\frac{\partial \pi_1}{\partial y_1} = x_1 \frac{\partial p}{\partial y_1} - \frac{\partial K_1^y}{\partial y_1} = 0
\]

and

\[
\frac{\partial \pi_2}{\partial y_2} = x_2 \frac{\partial p}{\partial y_2} - \frac{\partial K_2^y}{\partial y_2} = 0.
\]

Denote by \( y_1^{|y_2=0} \) (\( y_2^{|y_1=0} \)) the solution of \( \frac{\partial \pi_1}{\partial y_1} = 0 \) (\( \frac{\partial \pi_2}{\partial y_2} = 0 \)) given a certain value of \( y_2 \) (\( y_1 \)). Then \( y_1^{|y_2=0} \) and \( y_2^{|y_1=0} \) solve

\[
\left[ x_1 \frac{\partial p}{\partial y_1} - \frac{\partial K_1^y}{\partial y_1} \right]_{y_2=0} = 0
\]

and

\[
\left[ x_2 \frac{\partial p}{\partial y_2} - \frac{\partial K_2^y}{\partial y_2} \right]_{y_1=0} = 0.
\]

Assume (without loss of generality) that \( y_1^{|y_2=0} > y_2^{|y_1=0} \).

With constant marginal costs (Case 1 and 2 in Proposition 3) it follows

\[
\left[ \frac{\partial p}{\partial y_1} \right]_{\{ y_1^* \}_{y_2=0,y_2=0}} = \frac{\partial K_1^y}{\partial y_1} \frac{1}{x_1}
\]

and

\[
\left[ \frac{\partial p}{\partial y_2} \right]_{\{ y_2^* \}_{y_1=0,y_1=0}} = \frac{\partial K_2^y}{\partial y_2} \frac{1}{x_2}.
\]

\( y_1^{|y_2=0} > y_2^{|y_1=0} \) and \( \frac{\partial^2 p}{\partial y_2^2} \leq 0 \) leads to
\[ \frac{\partial p}{\partial y_1} \|_{y_1^* | y_2 = 0, y_2 = 0} \leq \frac{\partial p}{\partial y_2} \|_{y_1^* | y_2 = 0, y_1 = 0} \]

and so

\[ \frac{\partial K^y}{\partial y_1} \bigg|_{x_1} \leq \frac{\partial K^y}{\partial y_2} \bigg|_{x_2}. \]

Hence, it follows

\[ \frac{\partial p}{\partial y_2} \|_{y_1^* | y_2 = 0, y_2 = 0} = \frac{\partial p}{\partial y_1} \|_{y_1^* | y_2 = 0, y_2 = 0} = \frac{\partial K^y}{\partial y_1} \bigg|_{x_1} \leq \frac{\partial K^y}{\partial y_2} \bigg|_{x_2} \]

and so

\[ \frac{\partial \pi_2}{\partial y_2} \|_{y_1^* | y_2 = 0, y_2 = 0} = \left[ x_2 \frac{\partial p}{\partial y_2} - \frac{\partial K^y}{\partial y_2} \right] \|_{y_1^* | y_2 = 0, y_2 = 0} \leq 0. \]

This leads to

\[ \frac{\partial \pi_2}{\partial y_2} \|_{y_1^* | y_2 = 0, y_2 = 0} = \left[ x_2 \frac{\partial p}{\partial y_2} - \frac{\partial K^y}{\partial y_2} \right] \|_{y_1^* | y_2 = 0, y_2 = 0} \leq 0. \] (38)

With dependent marginal costs (Case 3 in Proposition 3), \( y_1^* \|_{y_2 = 0} > y_2^* \|_{y_1 = 0} \) and \( \frac{\partial^2 K^y}{\partial y_1 \partial y_2} \|_{y_2 = 0} \geq 0 \) it follows

\[ \frac{\partial K^y}{\partial y_1} \|_{y_1^* | y_2 = 0, y_2 = 0} \geq \frac{\partial K^y}{\partial y_2} \|_{y_1^* | y_1 = 0, y_1 = 0} \cdot \]

Hence

\[ x_1^* \frac{\partial p}{\partial y_1} \|_{y_1^* | y_2 = 0, y_2 = 0} \geq x_2^* \frac{\partial p}{\partial y_2} \|_{y_1^* | y_2 = 0, y_1 = 0} \cdot \]

Because of

\[ \frac{\partial K^y}{\partial y_1} \|_{Y} = \frac{\partial K^y}{\partial y_2} \|_{Y} \forall y_1, y_2 \]

it follows

\[ \frac{\partial \pi_2}{\partial y_2} \|_{y_1^* | y_2 = 0, y_2 = 0} = \left[ x_2^* \frac{\partial p}{\partial y_2} - \frac{\partial K^y}{\partial y_2} \right] \|_{y_1^* | y_2 = 0, y_2 = 0} \leq 0. \]

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This leads to
\[
\frac{\partial \pi_2}{\partial y_2} |_{y_1^* | y_2=0, y_2 \geq 0} = [x_2 \frac{\partial p}{\partial y_2} - \frac{\partial K^y_2}{\partial y_2}] |_{y_1^* | y_2=0, y_2 \geq 0} \leq 0.
\] (39)

Equations 38 and 39 show that \( \{(x_1^*, y_1^* | y_2=0), (x_2^*, y_2^* = 0)\} \) with \( y_1^* | y_2=0 > y_2^* | y_1=0 \) is indeed a Nash Equilibrium.

2. Is \( \{(x_1^*, y_1^* | y_2=0), (x_2^*, y_2^* = 0)\} \) with \( y_1^* | y_2=0 > y_2^* | y_1=0 \) the unique Nash Equilibrium?

First, let us look how \( y_1^* \) reacts given one knows that \( y_2 \) changes:

By total differentiation of \( \frac{\partial \pi_1}{\partial y_1} = x_1 \frac{\partial p}{\partial y_1} - \frac{\partial K^y_1}{\partial y_1} \) = 0 one gets
\[
[x_1 \frac{\partial^2 p}{\partial y^2} - \frac{\partial^2 K^y_1}{\partial y^2} | dy_1 + [x_1 \frac{\partial^2 p}{\partial y^2} - \frac{\partial^2 K^y_1}{\partial y_1 \partial y_2} | dy_2 = 0.
\]

Because of
\[
\frac{\partial^2 K^y_1}{\partial y_1^2} = \frac{\partial^2 K^y_1}{\partial y_1 \partial y_2}
\]

it is true that
\[
dy_1 = -dy_2.
\]

One can easily do the same exercise with \( y_2^* \) and \( y_1 \) which delivers the equivalent result.

Hence, we have a 1:1 relationship and therefore perfect strategic substitutes.

Now we are going to rule out all other possible equilibria.
In a Nash Equilibrium with \( y_1^* < \bar{y}_1 \) it must be true that
\[
\frac{\partial \pi_1}{\partial y_1} = x_1 \frac{\partial p}{\partial y_1} - \frac{\partial K^y_1}{\partial y_1} \leq 0.
\]

Otherwise firm 1 has an incentive to produce more \( y_1 \). Because \( y_2^* | y_1=0 \) solves
\[
[x_2 \frac{\partial p}{\partial y_2} - \frac{\partial K^y_2}{\partial y_2} |_{y_1=0} = 0
\]
firm 2 will never produce more than \( y^*_2 \mid y_1 = 0 \), so that
\[
y^\text{max}_2 = y^*_2 \mid y_1 = 0.
\]
This leads to the fact that firm 1 at least produces
\[
y^\text{min}_1 = y^*_1 \mid y_2 = 0 - y^\text{max}_2.
\]
But this leads once again to the fact that firm 2 produces maximal
\[
y^\text{max}'_2 = y^\text{max}_2 - y^\text{min}_1.
\]
Knowing this, firm 1 will at least produce
\[
y^\text{min}'_1 = y^*_1 \mid y_2 = 0 - y^\text{max}'_2 > y^\text{min}_1.
\]
Continuing, one sees that \( y^\text{min}_1 \) converges to \( y^*_1 \mid y_2 = 0 \) and \( y^\text{max}_2 \) converges to 0.

Hence, \( \{ (x^*_1, y^*_1 \mid y_2 = 0), (x^*_2, y^*_2 = 0) \} \) with \( y^*_1 \mid y_2 = 0 > y^*_2 \mid y_1 = 0 \) is the unique Nash Equilibrium.

Q.E.D.

**Proof of Proposition 4:**

The profit function of firm \( i \) can be written as:

\[
\pi_i = \pi_i(x^*_i(X_{-i}, Y_{-i}), y^*_i(X_{-i}, Y_{-i}), X_{-i}, Y_{-i}).
\]

Total differentiation leads to

\[
d\pi_i = \left[ \frac{\partial \pi_i}{\partial x_i} \frac{\partial x^*_i}{\partial X_{-i}} + \frac{\partial \pi_i}{\partial y_i} \frac{\partial y^*_i}{\partial X_{-i}} \right] dX_{-i} + \left[ \frac{\partial \pi_i}{\partial x_i} \frac{\partial x^*_i}{\partial Y_{-i}} + \frac{\partial \pi_i}{\partial y_i} \frac{\partial y^*_i}{\partial Y_{-i}} \right] dY_{-i}.
\]

Therefore, one can distinguish between first and second order effects. The first order effects directly influence the profit whereby the second order effects work over changing the production decision of firm \( i \). If, before the shock comes in, the
firm was in a optimum ($\frac{\partial\pi_i}{\partial x_i} = 0$ and $\frac{\partial\pi_i}{\partial y_i} = 0$) and the shock is small then the second order effects are zero.

Hence, there are only first order effects:

$$d\pi_i = \frac{\partial\pi_i}{\partial X_{-i}} dX_{-i} + \frac{\partial\pi_i}{\partial Y_{-i}} dY_{-i}.$$  \hspace{1cm} (42)

This leads to

$$\pi_i = x_i * p(x_i, y_i, X_{-i}, Y_{-i}) - K(x_i) - K(y_i)$$ \hspace{1cm} (43)

and hence

$$d\pi_i = x_i * \frac{\partial p}{\partial X_{-i}} dX_{-i} + x_i * \frac{\partial p}{\partial Y_{-i}} dY_{-i}.$$ \hspace{1cm} (44)

Therefore, it gets obvious that the change of the price determines the effect on the profit.

Q.E.D.

**Proof of Proposition 6:**

For the proof we proceed in two steps: First we show that $\frac{\partial x_i}{\partial y_G} > 0$ and afterwards proof that $\frac{\partial y_i}{\partial y_G} > 0$.

1.

Because of symmetry and uniqueness of the equilibrium using the implicit function theorem leads to:

$$\frac{\partial x_i}{\partial y_G} = \frac{\partial x_1}{\partial y_G} = \frac{|D_x|}{|D|}$$

with

$$|D_x| = \begin{vmatrix}
-\partial f^1/\partial y_G & \partial f^1/\partial y_1 & \partial f^1/\partial y_2 & \partial f^1/\partial x_2 & \partial f^1/\partial x_3 & \partial f^1/\partial y_3 & \cdots & \partial f^1/\partial y_{N-1} & \partial f^1/\partial y_N & \partial f^1/\partial x_N & \partial f^1/\partial y_N
-\partial f^2/\partial y_G & \partial f^2/\partial y_1 & \partial f^2/\partial y_2 & \partial f^2/\partial x_2 & \partial f^2/\partial x_3 & \partial f^2/\partial y_3 & \cdots & \partial f^2/\partial y_{N-1} & \partial f^2/\partial y_N & \partial f^2/\partial x_N & \partial f^2/\partial y_N
-\partial f^3/\partial y_G & \partial f^3/\partial y_1 & \partial f^3/\partial y_2 & \partial f^3/\partial x_2 & \partial f^3/\partial x_3 & \partial f^3/\partial y_3 & \cdots & \partial f^3/\partial y_{N-1} & \partial f^3/\partial y_N & \partial f^3/\partial x_N & \partial f^3/\partial y_N
-\partial f^4/\partial y_G & \partial f^4/\partial y_1 & \partial f^4/\partial y_2 & \partial f^4/\partial x_2 & \partial f^4/\partial x_3 & \partial f^4/\partial y_3 & \cdots & \partial f^4/\partial y_{N-1} & \partial f^4/\partial y_N & \partial f^4/\partial x_N & \partial f^4/\partial y_N
-\partial f^N/\partial y_G & \partial f^N/\partial y_1 & \partial f^N/\partial y_2 & \partial f^N/\partial x_2 & \partial f^N/\partial x_3 & \partial f^N/\partial y_3 & \cdots & \partial f^N/\partial y_{N-1} & \partial f^N/\partial y_N & \partial f^N/\partial x_N & \partial f^N/\partial y_N
\end{vmatrix}$$

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\[ |D| = \]

| \(f^1_{x_1}\) | \(f^1_{y_1}\) | \(f^1_{x_2}\) | \(f^1_{y_2}\) | \(f^1_{x_3}\) | \(f^1_{y_3}\) | \ldots | \(f^1_{x_{N-1}}\) | \(f^1_{y_{N-1}}\) | \(f^1_{x_N}\) | \(f^1_{y_N}\) |
| \(f^2_{x_1}\) | \(f^2_{y_1}\) | \(f^2_{x_2}\) | \(f^2_{y_2}\) | \(f^2_{x_3}\) | \(f^2_{y_3}\) | \ldots | \(f^2_{x_{N-1}}\) | \(f^2_{y_{N-1}}\) | \(f^2_{x_N}\) | \(f^2_{y_N}\) |
| \(f^3_{x_1}\) | \(f^3_{y_1}\) | \(f^3_{x_2}\) | \(f^3_{y_2}\) | \(f^3_{x_3}\) | \(f^3_{y_3}\) | \ldots | \(f^3_{x_{N-1}}\) | \(f^3_{y_{N-1}}\) | \(f^3_{x_N}\) | \(f^3_{y_N}\) |
| \(f^4_{x_1}\) | \(f^4_{y_1}\) | \(f^4_{x_2}\) | \(f^4_{y_2}\) | \(f^4_{x_3}\) | \(f^4_{y_3}\) | \ldots | \(f^4_{x_{N-1}}\) | \(f^4_{y_{N-1}}\) | \(f^4_{x_N}\) | \(f^4_{y_N}\) |

| \(f^N_{x_1}\) | \(f^N_{y_1}\) | \(f^N_{x_2}\) | \(f^N_{y_2}\) | \(f^N_{x_3}\) | \(f^N_{y_3}\) | \ldots | \(f^N_{x_{N-1}}\) | \(f^N_{y_{N-1}}\) | \(f^N_{x_N}\) | \(f^N_{y_N}\) |

For the proof that
\[ \frac{\partial x_i}{\partial y_G} = \frac{|D_x|}{|D|} > 0 \]
we show that the numerator and denominator is positive.

1.1 \(|D_x| > 0\)

With the conditions form Proposition 6 one can substitute:

\[ \frac{\partial f^z}{\partial y_G} = -a \forall z \in \{1, 3, 5, \ldots, N-1\} \]

\[ \frac{\partial f^z}{\partial y_w} = a \forall z \in \{1, 3, 5, \ldots, N-1\} \lor \forall w \in \{1, 2, 3, \ldots, N\} \]

\[ \frac{\partial f^z}{\partial x_w} = c \forall z \in \{3, 5, \ldots, N-1\} \lor \forall w \in \{1, 2, 3, \ldots, N\} \setminus z \]

\[ \frac{\partial f^z}{\partial x_{(z+1)/2}} = c \forall z \in \{3, 5, \ldots, N-1\} \]

\[ \frac{\partial f^z}{\partial y_G} = 0 \forall z \in \{2, 4, 6, \ldots, N\} \]

\[ \frac{\partial f^z}{\partial y_{z/2}} = b \forall z \in \{2, 4, 6, \ldots, N\} \]

\[ \frac{\partial f^z}{\partial y_w} = d \forall z \in \{2, 4, 6, \ldots, N\} \lor \forall w \in \{1, 2, 3, \ldots, N\} \setminus z/2 \]

\[ \frac{\partial f^z}{\partial y_G} = 0 \forall z \in \{2, 4, 6, \ldots, N\} \]
\[
\frac{\partial f^z}{\partial y_w} = 0 \forall z \in \{2, 4, 6, ..., N\} \lor w \in \{1, 2, 3, ..., N\} \setminus \{z/2\}
\]

\[
\begin{array}{cccccccccccc}
-a & a & c & a & c & a & \ldots & \ldots & c & a & c & a \\
0 & b & d & 0 & d & 0 & \ldots & \ldots & d & 0 & d & 0 \\
-a & a & e & a & c & a & \ldots & \ldots & c & a & c & a \\
0 & 0 & a & b & d & 0 & \ldots & \ldots & d & 0 & d & 0 \\
-a & a & c & a & e & a & \ldots & \ldots & c & a & c & a \\
0 & 0 & d & 0 & a & b & \ldots & \ldots & d & 0 & d & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\
-a & a & c & a & c & a & \ldots & \ldots & c & a & e & a \\
0 & 0 & d & 0 & d & 0 & \ldots & \ldots & d & 0 & a & b \\
\end{array}
\]

Through subtracting line 1 from all lines \(z\) with \(z \in \{3, 5, 7, ..., N-1\}\) one gets:

\[
\begin{array}{cccccccccccc}
-a & a & c & a & c & a & \ldots & \ldots & c & a & c & a \\
0 & b & d & 0 & d & 0 & \ldots & \ldots & d & 0 & d & 0 \\
0 & 0 & e - c & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & a & b & d & 0 & \ldots & \ldots & d & 0 & d & 0 \\
0 & 0 & 0 & 0 & e - c & 0 & \ldots & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & d & 0 & a & b & \ldots & \ldots & d & 0 & d & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & \ldots & e - c & 0 & 0 & 0 \\
0 & 0 & d & 0 & d & 0 & \ldots & \ldots & a & b & d & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 & e - c & 0 \\
0 & 0 & d & 0 & d & 0 & \ldots & \ldots & d & 0 & a & b \\
\end{array}
\]

One sees that there exists a line \(z \in \{3, 5, 7, ..., N-1\}\) where only the element in the \(z\) row is not zero. One can therefore use these lines to derive an upper triangle matrix:

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The Determinant is

\[ |D_x| = -ab^N(e - c)^{N-1} \]

Resubstituting gives

if N even:

\[
|D_x| = - \left[ \frac{\partial p}{\partial y} + x_i \frac{\partial^2 p}{\partial y^2} \right] [x_i] \left[ \frac{\partial^2 p}{\partial y^2} - \frac{\partial^2 K_i(y_i)}{\partial y^2} \right] \left[ \frac{\partial p}{\partial x} - \frac{\partial K_i(x_i)}{\partial x} \right]_{N-1} > 0
\]

if N uneven:

\[
|D_x| = - \left[ \frac{\partial p}{\partial y} + x_i \frac{\partial^2 p}{\partial y^2} \right] [x_i] \left[ \frac{\partial^2 p}{\partial y^2} - \frac{\partial^2 K_i(y_i)}{\partial y^2} \right] \left[ \frac{\partial p}{\partial x} - \frac{\partial K_i(x_i)}{\partial x} \right]_{N-1} > 0
\]

Therefore if follows \(|D| > 0 \forall N.\)

1.2 \(|D| > 0\)

\[
|D| = 
\]

\[
\begin{vmatrix}
-a & a & c & a & c & a & ... & c & a & c & a \\
0 & b & 0 & 0 & d & 0 & ... & d & 0 & d & 0 \\
0 & 0 & e - c & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & b & d & 0 & ... & d & 0 & d & 0 \\
0 & 0 & 0 & 0 & e - c & 0 & ... & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & b & ... & d & 0 & d & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & ... & 0 & e - c & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 & b \\
0 & 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\
\end{vmatrix}
\]

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We know that this matrix is diagonally dominant. Furthermore if a matrix is diagonal dominant and its diagonal elements are positive then the matrix is positive definite. On the diagonal there are the second-order conditions that are all negative. Through multiplying every line with \((-1)\) one gets positive diagonal elements and through the odd number of lines a scalar of +1. This gives a positive definite matrix and therefore the determinant has to be positive.

2.

Total derivative of \(\frac{\partial \pi_i}{\partial y_i}\):

\[
\left[ \frac{\partial p}{\partial y_i} + x_i \frac{\partial^2 p}{\partial x_i \partial y_i} \right] dx_i + \left[ x_i \frac{\partial^2 p}{\partial y_i^2} - \frac{\partial^2 K_i(y_i)}{\partial y_i^2} \right] dy_i
\]

One sees that the direct effect of \(dY_{-i}\) is zero and the sum of the direct effect of \(dX_{-i}\) and the indirect effect through \(dx_i\) is positive through the technical assumption in the proposition. Therefore \(y_i\) has to increase to ensure that \(\frac{\partial \pi_i}{\partial y_i}\) remains zero.

Q.E.D.

Proof of Lemma 1:

The existence of a Nash-Equilibrium in pure strategies follows immediately from Proposition 1.

For the uniqueness we apply the contraction mapping principle. Beforehand, we reduce the strategy space form \(R^2\) to \(R\) because \(y_i\) is directly determined through \(x_i\). To see this we write down the FOC with respect to the public good

\[
\frac{\partial \pi_i}{\partial y_i} = cx_i - 2fy_i = 0.
\]
Rewriting leads to

\[ y_i = \frac{c}{2f} x_i. \]

Plugging back into the profit function of firm \( i \) this leads to

\[ \pi_i = x_i(A - bx_i - b \sum_{j=1; j \neq i}^{N} x_j + c \frac{c}{2f} x_i + c \sum_{j=1; j \neq i}^{N} \frac{c}{2f} x_j) - dx_i^2 - f\left(\frac{c}{2f} x_i\right)^2. \]

The first order conditions of the \( N \) firms with respect to the private good are a contraction mapping if

\[ \sum_{j=1; j \neq i}^{N} \left| \frac{\partial^2 \pi_i}{\partial x_i \partial x_j} \right| < \left| \frac{\partial^2 \pi_i}{\partial x_i^2} \right| \quad \forall i \in \{1, ..., N\} \]

\[
(N - 1) \left| -b + \frac{c^2}{2f} \right| < \left| -2b - 2d + \frac{c^2}{2f} \right|
\]

If \( f \geq \frac{c^2}{2b} \) then \(-b + \frac{c^2}{2f} \leq 0\).

Therefore:

\[ (1 - N)(-b + \frac{c^2}{2f}) < 2b + 2d - \frac{c^2}{2f} \]

\[ N < \frac{3b + 2d - \frac{c^2}{2f}}{b - \frac{c^2}{2f}} \]

If \( f < \frac{c^2}{2b} \) then \(-b + \frac{c^2}{2f} > 0\).

Therefore:

\[ (N - 1)(-b + \frac{c^2}{2f}) < 2 + 2d - \frac{1}{2f} \]

\[ N < \frac{b + 2d}{\frac{c^2}{2f} - b} \]

Q.E.D.

Proof of Proposition 8:
For the first part of the proof we take the derivative of $X$ with respect to $N$ and look at the sign.

$$\frac{\partial X}{\partial N} = \frac{\partial [N \ast x_i(N)]}{\partial N} = N \ast \frac{\partial x_i}{\partial N} + x_i$$  \hspace{1cm} (46)$$

$$\frac{\partial X}{\partial N} = -N \ast \frac{A}{[b(1 + N) - \frac{1}{2} c^2 N + 2d]^2} \ast (b - \frac{1}{2} c^2 f) + \frac{A}{b(1 + N) - \frac{1}{2} c^2 N + 2d}$$  \hspace{1cm} (47)$$

$$\frac{\partial X}{\partial N} = \frac{A}{b(1 + N) - \frac{1}{2} c^2 N + 2d}(-N \ast \frac{1}{b(1 + N) - \frac{1}{2} c^2 N + 2d} \ast (b - \frac{1}{2} c^2 f) + 1)$$  \hspace{1cm} (48)$$

$$\frac{\partial X}{\partial N} = \frac{A \ast [b + 2d]}{[b(1 + N) - \frac{1}{2} c^2 N + 2d]^2} > 0$$  \hspace{1cm} (49)$$

ByLemma 2 we know that the private and the public good are always individually produced in the same ratio. Summing up, we see that the total production must have the same ratio. If $X$ is always increasing, then $Y$ is increasing too.

Q.E.D.

**Proof of Proposition 9:**

For the proof we use the fact that $y_i = \frac{c^2}{2f} x_i$ (22) and rewrite the demand function (16) as follows:

$$p = A - bX + cY = A - bX(N) + c\frac{c}{2f} X(N) = A + X(N)(-b + \frac{c^2}{2f})$$  \hspace{1cm} (50)$$

$$p = A + N \ast \frac{A}{b(1 + N) - \frac{1}{2} c^2 N + 2d}(-b + \frac{c^2}{2f})$$  \hspace{1cm} (51)$$

Now we can take the first derivative of $p$ with respect to $N$ and look at the sign:

$$\frac{\partial p}{\partial N} = (-b + \frac{c^2}{2f}) \ast \frac{A(b + 2d)}{[b(1 + N) - \frac{1}{2} c^2 N + 2d]^2}$$  \hspace{1cm} (52)$$

Therefore

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\[ \frac{\partial p}{\partial N} < 0 \text{ if } f > \frac{c^2}{2b} \]
\[ \frac{\partial p}{\partial N} = 0 \text{ if } f = \frac{c^2}{2b} \rightarrow p = A \]
\[ \frac{\partial p}{\partial N} > 0 \text{ if } f < \frac{c^2}{2b} \]

Q.E.D.

Proof of Proposition 10:

\[ \pi_i = p * x_i - dx^2 - f\left[\frac{1}{2f}x_i\right]^2 = p * x_i - x_i^2(d + \frac{1}{4f}) \] (53)

\[ \pi_i = \frac{A^2(b + d - \frac{c^2}{2f})}{(b(1 + N) - \frac{c^2}{2f}N + 2d)^2} \] (54)

\[ \frac{\partial \pi_i}{\partial N} = \frac{A^2(b - \frac{c^2}{2f})(-2b - 2d + \frac{c^2}{2f})}{\left(b(1 + N) - \frac{c^2}{2f}N + 2d\right)^3} \] (55)

The last term \((-2b - 2d + \frac{c^2}{2f})\) is always negative because of the second order conditions of the maximization problem. We know by Assumption A7 that it must be true that

\[ c^2 < (-2b - 2d)(-2f) \] (56)

Solving for \(f\) yields to

\[ f > \frac{c^2}{4b + 4d} \] (57)

Assuming \(-2b - 2d + \frac{c^2}{2f} < 0\) and solving for \(f\) leads to:

\[ f > \frac{c^2}{4b + 4d} \] (58)

Therefore we have to look at \(b - \frac{c^2}{2f}\) and at the denominator.

Case 1: \( f = \frac{c^2}{2b} \)
Then \( b - \frac{c^2}{2f} = 0 \) and therefore \( \frac{\partial \pi}{\partial N} = 0 \).

**Case 2:** \( f > \frac{c^2}{2b} \)

Then \( b - \frac{c^2}{2f} > 0 \) and the sign of \( \frac{\partial \pi}{\partial N} \) depends on \( N \) because of the term \( b(1+N) - \frac{c^2}{2f}N + 2d \).

This term is always positive:

\[
\begin{align*}
  b(1+N) - \frac{c^2}{2f}N + 2d &= b + N(b - \frac{c^2}{2f}) + 2d \\
  \text{(59)}
\end{align*}
\]

Therefore \( \frac{\partial \pi}{\partial N} < 0 \) for all \( N > 0 \)

**Case 3:** \( f < \frac{c^2}{2b} \)

Then \( b - \frac{c^2}{2f} < 0 \) and the sign of \( \frac{\partial \pi}{\partial N} \) depends on \( N \) because of the term \( b(1+N) - \frac{c^2}{2f}N + 2d \).

This term is zero if

\[
\begin{align*}
  N^* &= \frac{2d + b}{\frac{c^2}{2f} - b} > 0 \\
  \text{(60)}
\end{align*}
\]

The slope of \( b(1+N) - \frac{c^2}{2f}N + 2d \) with respect to \( N \) is

\[
\begin{align*}
  \frac{\partial (b(1+N) - \frac{c^2}{2f}N + 2d)}{\partial N} &= b - \frac{c^2}{2f} < 0 \\
  \text{(61)}
\end{align*}
\]

Therefore \( \frac{\partial \pi}{\partial N} > 0 \) in the relevant area where \( N < N^* \).

Q.E.D.

**Proof of Proposition 11:**

First we calculate the consumer surplus.

\[
\begin{align*}
  CS &= (A + cY - p) \times X \times 0.5 = 0.5 \times bX^2 \\
  \text{(62)}
\end{align*}
\]

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The total surplus as the sum of the firms’ profits and the consumers’ surplus.

\[ TS = N \pi_i + CS \]  

(63)

To see the reaction of the total surplus caused by a variation in \( N \), we take the first derivative of \( TS \) with respect to \( N \) and look at the sign.

\[ \frac{\partial TS}{\partial N} = \frac{\partial (N \pi_i)}{\partial N} + \frac{\partial CS}{\partial N} \]  

(64)

\[ \frac{dT S}{dN} = A^2 bN \left( d + \frac{c^2}{47} \right) + (b + d - \frac{c^2}{47})(b + .5c^2N + 2d) \frac{bN + .5c^2N + 2d^3}{b(1+N) + .5c^2N + 2d^3} > 0 \]  

(65)

Q.E.D.
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References


